WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 1st Semester Examination, 2022-23

MTMACOR02T-MATHEMATICS (CC2)

ALGEBRA

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: $2\times5 = 10$

(a) If a, b, c, d are positive real numbers, not all equal, prove that

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)>16$$

(b) Prove that $\sqrt[n]{i}+\sqrt[n]{-i}=2\cos\frac{\pi}{2n}$, where n is a positive integer greater than 1 and $\sqrt[n]{z}$ is the principal nth root of z.

(c) Apply Descartes' rule of sign to find the least number of non real roots of the equation $x^{10}-x^3=0$.

(d) If a and b are integers s.t. g.c.d.$(a, b)=1$ then prove that g.c.d$(a+b, a\cdot b)=1$.

(e) Show that the product of any four consecutive integers is divisible by 24.

(f) Give an example of a surjective mapping $f: S\to S$ which is not injective, where S is an infinite set.

(g) Find a relation on the set of positive integers which is transitive but neither reflexive nor symmetric.

(h) Solve the equation $2x^3-x^2-18x+9=0$ if two of its roots are equal in magnitude but opposite in sign.

(i) Give an example of a 3×3 matrix whose eigenvalues are 1, 2 and 3.

2. (a) If $3s=a+b+c+d$, where a, b, c, d and $s-a$, $s-b$, $s-c$, $s-d$ are all positive, prove that 4

$$abcd>81(s-a)(s-b)(s-c)(s-d),$$

unless $a=b=c=d$.

(b) If a, b, c are positive real numbers s.t. $a+b+c=1$, then prove that 4

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2+\left(c+\frac{1}{c}\right)^2\geq 33\frac{1}{3}$$

3. (a) If α be a special root of the equation $x^{12}-1=0$, prove that 4

$$(\alpha+\alpha^{11})(\alpha^5+\alpha^7)=-3$$

(b) Solve the equation $x^4-6x^2-16x-15=0$ by Ferrari's method. 4

4. (a) Show that the principal value of the ratio of $(1+i)^{1-i}$ and $(1-i)^{1+i}$ is $\sin(\log 2)+i\cos(\log 2)$. 4

(b) If $\cos\alpha+\cos\beta+\cos\gamma=0=\sin\alpha+\sin\beta+\sin\gamma$ then show that 4

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\frac{3}{2}=\sin^2\alpha+\sin^2\beta+\sin^2\gamma.$$

5. (a) Prove that for all $n\in\mathbb{N}$, $(2+\sqrt{3})^n+(2-\sqrt{3})^n$ is an even integer. 4

(b) Prove that $13^{73}+14^3\equiv 2\,(\text{mod } 11)$. 4

6. (a) Examine whether the relation ρ is an equivalence relation on the set $\mathbb{Z}$ of all integers, where 4

$$\rho=\{(m,n)\in\mathbb{Z}\times\mathbb{Z}:|m-n|\le 3\}.$$

(b) Suppose that $f:A\to B$ is a function. Show that f is $1-1$ if and only if there exists an onto function $g:B\to A$ satisfying $g(f(a))=a, \forall a\in A$. 2+2

7. (a) Prove that the eigenvalues of a real symmetric matrix are all real. 4

(b) Compute the inverse of the following matrix by row transformations: 4

$$A=\begin{pmatrix}3 & 1 & 1\\ 4 & 2 & -1\\ 7 & 3 & 1\end{pmatrix}.$$

8. (a) Determine the conditions for which the system of equation has 4
(i) only one solution (ii) no solution (iii) many solutions

$$x+y+z=1$$
$$x+2y-z=b$$
$$5x+7y+az=b^2$$

(b) Reduce the matrix 4

$$A=\begin{pmatrix}1 & 2 & -2\\ 2 & 1 & 2\\ -2 & 2 & 1\end{pmatrix}$$

to a row-reduced Echelon form and find its rank.

9. (a) Show that the eigen vectors corresponding to distinct eigenvalues of an $n\times n$ matrix A are linearly independent. 4

(b) Find the eigenvalues and the corresponding eigen vectors of the following matrix 4

$$\begin{bmatrix}2 & 2 & 1\\ 1 & 3 & 1\\ 1 & 2 & 2\end{bmatrix}$$

10. (a) Show that zero is a characteristic root of a matrix A if and only if A is singular. 3

(b) Verify Cayley-Hamilton theorem for the matrix 5

$$A=\begin{pmatrix}1 & 2 & 1\\ 1 & -1 & 1\\ 2 & 3 & -1\end{pmatrix}$$

Express A^{-1} as a polynomial in A and then compute A^{-1}.

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